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A new analytical method to solve the heat equation for a multi-dimensional composite slab

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Abstract
A novel analytical approach has been developed for heat conduction in a multi-dimensional composite slab subject to time-dependent boundary changes of the first kind. Boundary temperatures are represented as Fourier series. Taking advantage of the periodic properties of boundary changes, the analytical solution is obtained and expressed explicitly. Nearly all the published works necessitate searching for associated eigenvalues in solving such a problem even for a one-dimensional composite slab. In this paper, the proposed method involves no iterative computation such as numerically searching for eigenvalues and no residue evaluation. The adopted method is simple which represents an extension of the novel analytical approach derived for the one-dimensional composite slab. Moreover, the method of '*separation of variables*' employed in this paper is new. The mathematical formula for solutions is concise and straightforward. The physical parameters are clearly shown in the formula. Further comparison with numerical calculations is presented.

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1. Introduction

Many processes are governed by multi-dimensional heat conduction equations for composite slabs in many areas of science and engineering ranging from building physics, thermodynamics, combustion, reacting flow processes, heat transfer, unconfined groundwater flows, and many others. Solution strategies have generally followed analytical and numerical

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directions. The first approach, though limited in terms of versatility of applications, has immense value in estimating material properties and validating numerical solutions. Since analytical heat conduction analysis in multi-dimensions is complex and demanding, practical guidelines for thermal field calculation are very few. Due to the highly complex nature of the problem, engineers and researchers often simplify actual complex problems to one-dimensional cases. Under particular conditions, it is relatively easier to solve one-dimensional conduction equations analytically. To date, a few reported results of temperature distribution or heat flux fields in multi-dimensional composite slabs have appeared in the literature. A number of standard textbooks [1, 2] have been mainly devoted to one-dimensional problems. Even for the one-dimensional case, the problem for the composite slab is very complicated [3]. In fact, one-dimensionally explicit solutions have been derived for only the three-layer slab recently [4]. For multi-dimensional problems, recent results were provided by de Monte [5] for two-dimensional and two-layer slabs. A 'natural' analytical approach derived for the one-dimensional composite slab was extended for the two-dimensional slab [6]. The associated eigenvalue problem was solved with numerical procedure-Newton iteration.

Classically, transient heat equations for a multi-dimensional single slab were solved with techniques such as the Green function [1], the orthogonal expansion [3] and Laplace transform [1]. For the multi-dimensional composite slab, these techniques are also often employed [7–9]. Associated eigenvalue problems are needed for the solution in the first two methods. Computations for the multi-dimensional composite slab exhibit a few special features. The eigenvalues may become imaginary, so the corresponding eigenfunctions will have imaginary arguments [9]. Moreover, attention must be paid when computing eigenvalues since the spacing between successive eigenvalues changes between zero and a maximal value [9]. For example, in the 'natural' analytical approach developed by de Monte [5], the associated eigenvalue problem for two-dimensional composites with two rectangular parallel layers was spit up into two one-dimensional eigenvalue problems. In the direction of the layers, the problem was a special case of the Sturm–Liouville problem. However, in the direction perpendicular to the layers, the problem was characterized by real and imaginary eigenvalues. Special care was then taken. Numerically, the imaginary eigenvalues can produce instability [9]. Hence, the associated eigenvalue problem can become too complicated to solve. Even for a one-dimensional problem, many workers have mentioned the complexity of the associated eigenvalue problem [10].

The third commonly adopted method is the Laplace transform which often yields residue computation. For the composite slab, the computation is found by directly and numerically searching for the roots of a hyperbolic equation, finding the derivatives of the equation, and evaluating and summing the residues. The calculation procedure is tedious if the slab has more than two layers [11], as numerical searching roots have to be made with very fine increment for inverse Laplace transforms to prevent missing roots which can lead to a wrong inverse.

These methods were also reviewed by de Monte [5]. He also pointed out that all the papers agree that the solution is able to deal only with homogeneous boundary conditions of the first and second kinds in the direction parallel to the layers, since the linear homogeneous boundary condition of the third kind unconditionally produces mathematical incompatibilities. More works are presented by Haji-Shaikh *et al* [9, 12], which employed the Green's function method.

All the above-cited works require solving of eigenvalue problems. Recently, a novel analytical method was developed to tackle one-dimensional transient heat problems for the composite slab subject to periodic temperature changes [13]. Taking the advantage of the periodic properties of boundary changes, the corresponding analytical solution was obtained and expressed explicitly. Unlike most of the traditional methods, the new method involves no

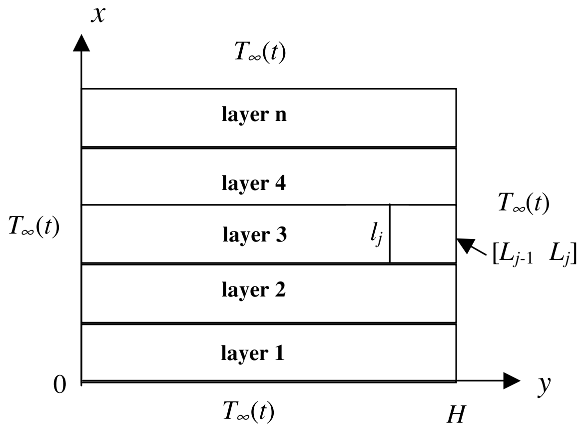

Figure 1. Schematic diagram of a two-dimensional composite slab studied in this paper.

residue evaluation and no iterative computation such as a numerical search for eigenvalues. The adopted method is simple and concise with high accuracy.

In this paper, the developed one-dimensional analytical method is extended to multi-dimensional geometry for the composite slab. The technique of '*separation of variables*' is used. As the boundary conditions are not homogeneous which exhibit a different feature than a usual way, a new separation technique is proposed.

Compared to the work reviewed above, firstly, boundary conditions or ambient temperature are given more generally with time dependence. Secondly, there is no need to numerically search for eigenvalues and to evaluate residues. And the analytical solution is concise and easy to apply. The physical parameters are clearly shown in the mathematical formula. Further comparison of the results demonstrates the high accuracy level of the developed analytical method.

2. Mathematical formulation

2.1. Governing equations

An n-layer composite slab has constant thermal conductivity, diffusivity and density for each layer whose thermal conductivity, diffusivity and thickness are presented as λ_j, k_j and l_j, $j = 1, \ldots, n$. The basic geometry considered here is a two-dimensional slab in x and y directions. So the layers have regional lengths in x direction as l_1, l_2 and l_n. Denote $L_j = l_1 + \cdots + l_j$, $j = 1, \ldots, n$, the layer boundaries are $[L_0 = 0, L_1]$, $[L_1, L_2]$ and $[L_{n-1}, L_n]$. The schematic figure is shown in figure 1.

The general heat conduction in the slab with first-kind boundary conditions can be described by the following equations for temperatures $T_j(t, x, y)$:

$$\frac{\partial T_j(t,x,y)}{\partial t} = k_j \frac{\partial^2 T_j(t,x,y)}{\partial x^2} + k_j \frac{\partial^2 T_j(t,x,y)}{\partial y^2}, \qquad x \in [L_{j-1}, L_j], \quad y \in [0,1], \quad j = 1, \ldots, n. \tag{2.1}$$

The boundary conditions are

$$T_1(t, L_0, y) = T_\infty(t), \qquad y \in [0,1], \tag{2.2a}$$

$$T_j(t, L_j, y) = T_{j+1}(t, L_j, y), \qquad y \in [0,1], \quad j = 1, \ldots, n-1, \tag{2.2b}$$

$$-\lambda_j \frac{\partial T_j}{\partial x}(t, L_j, y) = -\lambda_{j+1} \frac{\partial T_{j+1}}{\partial x}(t, L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{2.2c}$$

$$T_n(t, L_n, y) = T_\infty(t), \qquad y \in [0, 1], \tag{2.2d}$$

$$T_j(t, x, 0) = T_\infty(t), \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{2.2e}$$

$$T_j(t, x, 1) = T_\infty(t), \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{2.2f}$$

$$T_j(0, x, y) = 0, \qquad x \in [L_{j-1}, L_j], \quad y \in [0, 1], \quad j = 1, \ldots, n. \tag{2.2g}$$

Without losing generality, it is assumed that $H = 1$ and boundary temperature is assumed to be a simple periodic excitation $T_\infty(t) = \cos(\omega t + \varphi)$. Cases with more general time-dependent boundaries are given later. The initial temperature is set to 0 for the sake of calculation convenience.

Our first step in solving the equations is to reduce the two-dimensional case into several one-dimensional cases so that the available one-dimensional results can be applied. The method of '*separation of variables*' is adopted. Classically, the application of '*separation of variables*' requires that the equations be linear and homogeneous. Unfortunately, this is not true in equations (2.1), (2.2). The difficulty lies in the boundary equations (2.2*a*), (2.2*d*)–(2.2*f*). Therefore, the technique proposed in the following is different from those commonly reported in textbooks and journals.

In the following, if there is no danger of confusion we shall only write the simple forms of all the notation. For example T_j, $T_j(L_n)$ instead of $T_j(t, x, y)$ and $T_j(t, L_n, y)$.

2.2. Modification of the problems

First, we proceed the solution by assuming the complex form of the boundary temperature $U_\infty(t) = \exp(\mathrm{i}\omega t + \mathrm{i}\varphi)$. The corresponding solution of equations (2.1), (2.2) is denoted as $U_j(t, x, y)$. Clearly, the real part of the solution is the sought-after solution: $T_j(t, x, y) = \mathrm{Real}(U_j(t, x, y))$, $j = 1, \ldots, n$.

Equations (2.1), (2.2) are split up into two simpler subproblems such as

$$U_j(t, x, y) = U_j^1(t, x, y) + U_j^2(t, x, y), \tag{2.3}$$

where $U_j^1(t, x, y)$ and $U_j^2(t, x, y)$ satisfy the following systems of P1 and P2.

$$P1: \quad \frac{\partial U_j^1}{\partial t} = k_j \frac{\partial^2 U_j^1}{\partial x^2} + k_j \frac{\partial^2 U_j^1}{\partial y^2}, \qquad x \in [L_{j-1}, L_j], \quad y \in [0, 1], \quad j = 1, \ldots, n \tag{2.4a}$$

and boundary conditions are

$$U_1^1(t, L_0, y) = U_\infty(t), \qquad y \in [0, 1], \tag{2.4b}$$

$$U_j^1(t, L_j, y) = U_{j+1}^1(t, L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{2.4c}$$

$$-\lambda_j \frac{\partial U_j^1}{\partial x}(t, L_j, y) = -\lambda_{j+1} \frac{\partial U_{j+1}^1}{\partial x}(t, L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{2.4d}$$

$$U_n^1(t, L_n, y) = U_\infty(t), \qquad y \in [0, 1], \tag{2.4e}$$

$$U_j^1(t, x, 0) = 0, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{2.4f}$$

$$U_j^1(t,x,1)=0, \qquad x\in[L_{j-1},L_j], \quad j=1,\dots,n. \tag{2.4g}$$

$$U_j^1(0,x,y)=0, \qquad x\in[L_{j-1},L_j], \quad y\in[0,1], \quad j=1,\dots,n. \tag{2.4h}$$

$$P2: \quad \frac{\partial U_j^2}{\partial t}=k_j\frac{\partial^2 U_j^2}{\partial x^2}+k_j\frac{\partial^2 U_j^2}{\partial y^2}, \qquad x\in[L_{j-1},L_j], \quad y\in[0,1], \quad j=1,\dots,n \tag{2.5a}$$

and boundary conditions are

$$U_1^2(t,L_0,y)=0, \qquad y\in[0,1], \tag{2.5b}$$

$$U_j^2(t,L_j,y)=U_{j+1}^2(t,L_j,y), \qquad y\in[0,1], \quad j=1,\dots,n-1, \tag{2.5c}$$

$$-\lambda_j\frac{\partial U_j^2}{\partial x}(t,L_j,y)=-\lambda_{j+1}\frac{\partial U_{j+1}^2}{\partial x}(t,L_j,y), \qquad y\in[0,1], \quad j=1,\dots,n-1, \tag{2.5d}$$

$$U_n^2(t,L_n,y)=0, \qquad y\in[0,1], \tag{2.5e}$$

$$U_j^2(t,x,0)=U_\infty(t), \qquad x\in[L_{j-1},L_j], \quad j=1,\dots,n, \tag{2.5f}$$

$$U_j^2(t,x,1)=U_\infty(t), \qquad x\in[L_{j-1},L_j], \quad j=1,\dots,n, \tag{2.5g}$$

$$U_j^2(0,x,y)=0, \qquad x\in[L_{j-1},L_j], \quad y\in[0,1], \quad j=1,\dots,n. \tag{2.5h}$$

3. Analytical solutions for P1

3.1. Reduction to one-dimensional subproblems

Denote $b_m=\frac{2-2\cos(m\pi)}{m\pi}$, $m=1,\dots,\infty$, then we claim that the problem P1 can be split up into the following subproblems P1-1 and P1-2.

$$P1\text{-}1: \quad \frac{\partial X_{jm}}{\partial t}-k_j\frac{\partial^2 X_{jm}}{\partial x^2}+k_jm^2\pi^2X_{jm}=0, \qquad x\in[L_{j-1},L_j], \quad j=1,\dots,n, \quad m=1,\dots,\infty, \tag{3.1a}$$

with boundaries:

$$X_{1m}(t,L_0)=b_mU_\infty(t), \qquad m=1,\dots,\infty, \tag{3.1b}$$

$$X_{jm}(t,L_j)=X_{(j+1)m}(t,L_j), \qquad j=1,\dots,n-1, \quad m=1,\dots,\infty, \tag{3.1c}$$

$$-\lambda_j\frac{\partial X_{jm}}{\partial x}(t,L_j)=-\lambda_{j+1}\frac{\partial X_{(j+1)m}}{\partial x}(t,L_j), \qquad j=1,\dots,n-1, \quad m=1,\dots,\infty, \tag{3.1d}$$

$$X_{nm}(t,L_n)=b_mU_\infty(t), \qquad m=1,\dots,\infty, \tag{3.1e}$$

$$X_{jm}(t,x)=0, \qquad x\in[L_{j-1},L_j], \quad j=1,\dots,n, \quad m=1,\dots,\infty. \tag{3.1f}$$

$$P1\text{-}2: \quad Y_m''(y)+m^2\pi^2Y_m(y)=0, \qquad y\in[0,1], \quad m=1,\dots,\infty, \tag{3.2a}$$

with boundaries

$$Y_m(0)=0, \qquad Y_m(1)=0, \qquad m=1,\dots,\infty. \tag{3.2b}$$

The solution of P1 is expressed as

$$U_j^1(t,x,y)=\sum_{m=1}^{\infty}X_{jm}(t,x)Y_m(y). \tag{3.3}$$

3.2. Verification of the claim

Firstly, for the jth layer, the time and spatial variables are separated by assuming a product solution of the form

$$U_j^1(t,x,y) = X_j(t,x)Y_j(y). \tag{3.4}$$

Substituting this assumed form of the solution into the homogeneous equation (2.4*a*) gives

$$\frac{\frac{\partial X_j}{\partial t} - k_j \frac{\partial^2 X_j}{\partial x^2}}{X_j} = \frac{k_j Y_j''}{Y_j}. \tag{3.5}$$

Setting each side of the above equation equal to $-\mu_j^2$ gives

$$Y_j'' + \frac{\mu_j^2}{k_j} Y_j = 0. \tag{3.6}$$

The solution Y_j is straightforward and given as

$$Y_j(y) = A_j \sin\left(\frac{\mu_j}{\sqrt{k_j}} y\right) + B_j \cos\left(\frac{\mu_j}{\sqrt{k_j}} y\right). \tag{3.7}$$

To satisfy the boundary condition (2.4*f*), (2.4*g*) we get that

$$\frac{\mu_j}{\sqrt{k_j}} = m\pi \quad \text{or} \quad \mu_{jm} = m\pi\sqrt{k_j} \qquad \text{and}$$

$$Y_{jm}(y) = A_{jm}\sin(m\pi y), \quad m = 1, \ldots, \infty. \tag{3.8}$$

It is easily seen that $Y_{jm}(y)$ is independent of layers, thereby we re-write as

$$Y_m(y) = A_m \sin(m\pi y), \qquad m = 1, \ldots, \infty. \tag{3.9}$$

Equations (3.6) and (3.9) demonstrate that Y_m is the solution to the system P1-2. Therefore, a more general form of the linear combination of these solutions will satisfy the system P1:

$$U_j^1(t,x,y) = \sum_{m=1}^{\infty} X_{jm}(t,x)\sin(m\pi y). \tag{3.10}$$

Note the coefficient A_m in equation (3.9) is embedded in $X_{jm}(t,x)$ which will be determined from the boundary conditions.

Just for the sake of convenience, we shall omit writing $m = 1, \ldots, \infty$ in the following. Combination of equations (3.5) and (3.8) gives

$$\frac{\partial X_{jm}}{\partial t} - k_j \frac{\partial^2 X_{jm}}{\partial x^2} + \mu_{jm}^2 X_{jm} = \frac{\partial X_{jm}}{\partial t} - k_j \frac{\partial^2 X_{jm}}{\partial x^2} + k_j m^2\pi^2 X_{jm} = 0,$$

which is equation (3.1*a*) in P1-1.

The specification of boundary conditions is not so straightforward as that in problem P1-2. To evaluate, we observe that the boundary conditions (2.4*b*) and (2.4*e*) from P1 in combination of equation (3.10) determine that

$$U_\infty(t) = U_1^1(t, L_0, y) = \sum_{m=1}^{\infty} X_{1m}(t, L_0)\sin(m\pi y), \tag{3.11a}$$

$$U_\infty(t) = U_n^1(t, L_n, y) = \sum_{m=1}^{\infty} X_{nm}(t, L_n)\sin(m\pi y). \tag{3.11b}$$

Using orthogonal properties of $\sin(m\pi y)$, $\int_0^1 \sin(m\pi y)\sin(k\pi y)\,\mathrm{d}y = 0$, for $m \neq k$, we get

$$\int_0^1 U_\infty \sin(m\pi y)\,\mathrm{d}y = \int_0^1 \sum_{k=1}^{\infty} X_{1k}(t, L_0)\sin(k\pi y)\sin(m\pi y)\,\mathrm{d}y. \tag{3.12}$$

The left side is $\frac{b_m}{2}U_\infty$ and the right side equals $X_{1m}(t, L_0) = \frac{1}{2}$. Hence

$$X_{1m}(t, L_0) = b_m U_\infty \qquad \text{and} \qquad X_{nm}(t, L_n) = b_m U_\infty, \tag{3.13}$$

which are the boundary specifications in equations (3.1*b*) and (3.1*e*) for P1-1. The linear property of equation (3.10) ensures that boundary conditions (2.4*c*), (2.4*d*) in P1 be satisfied by assuming the boundaries (3.1*c*), (3.1*d*) in P1-1. So the claim that P1 can be spit up into P1-1 and P1-2 is verified. We now proceed with the solution for problem P1-1.

3.3. Analytical solution to problem P1-1

A similar type of problem, without convection term, was presented and solved in our companion paper [13]. It is helpful to review the method in sufficient detail in solving P1-1 for understanding. To emphasize, let us re-write the problem P1-1 and omit writing $m = 1, \ldots, \infty$ in the following:

$$P1\text{-}1: \quad \frac{\partial X_{jm}}{\partial t} - k_j \frac{\partial^2 X_{jm}}{\partial x^2} + k_j m^2\pi^2 X_{jm} = 0, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{3.14a}$$

with boundaries:

$$X_{1m}(t, L_0) = b_m U_\infty, \tag{3.14b}$$

$$X_{jm}(t, L_j) = X_{(j+1)m}(t, L_j), \qquad j = 1, \ldots, n-1, \tag{3.14c}$$

$$-\lambda_j \frac{\partial X_{jm}}{\partial x}(t, L_j) = -\lambda_{j+1}\frac{\partial X_{(j+1)m}}{\partial x}(t, L_j), \qquad j = 1, \ldots, n-1, \tag{3.14d}$$

$$X_{nm}(t, L_n) = b_m U_\infty, \tag{3.14e}$$

$$X_{jm}(0, x) = 0, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n. \tag{3.14f}$$

3.3.1. Laplace transformation of the equations. Applying Laplace transformation on (3.14*a*) gives

$$(s + k_j m^2\pi^2)\overline{X_{jm}}(s, x) = k_j \frac{\partial^2 \bar{X}_{jm}}{\partial x^2}(s, x), \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n \tag{3.15a}$$

with boundaries

$$\bar{X}_{1m}(s, L_0) = b_m \bar{U}_\infty, \tag{3.15b}$$

$$\bar{X}_{jm}(s, L_j) = \bar{X}_{(j+1)m}(s, L_j), \qquad j = 1, \ldots, n-1, \tag{3.15c}$$

$$-\lambda_j \frac{\partial \bar{X}_{jm}}{\partial x}(s, L_j) = -\lambda_{j+1}\frac{\partial \bar{X}_{(j+1)m}}{\partial x}(s, L_j), \qquad j = 1, \ldots, n-1, \tag{3.15d}$$

$$\bar{X}_{nm}(s, L_n) = b_m \bar{U}_\infty. \tag{3.15e}$$

A bar over a function $f(t)$ designates its Laplace transform on t (e.g. [10]):

$$\bar{f}(s) = \mathcal{L}(f(t)) = \int_0^\infty \exp(-s\tau) f(\tau)\,\mathrm{d}\tau. \tag{3.16a}$$

The Laplace transform of a convolution is given by

$$\mathcal{L}(f_1(t) * f_2(t)) = \bar{f}_1(s)\bar{f}_2(s) \qquad \text{where}$$
$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau. \tag{3.16b}$$

The solution of the differential equation (3.15*a*) is obtained as

$$\begin{aligned} \bar{X}_{jm} &= A_{jm}\sinh(q_{jm}(x - L_{j-1})) + B_{jm}\cosh(q_{jm}(x - L_{j-1})), \\ &\quad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \end{aligned} \tag{3.17}$$

where

$$q_{jm} = \sqrt{\frac{s}{k_j} + m^2\pi^2}.$$

Denote $\xi_{jm} = q_{jm} l_j$, $j = 1, \ldots, n$ and $h_j = \frac{\lambda_{j+1}}{\lambda_j}\sqrt{\frac{k_j}{k_{j+1}}}$, $j = 1, \ldots, n-1$, coefficients A_{jm} and B_{jm} in equation (3.17) are determined by the boundary conditions (3.15*b*)–(3.15*e*) as

$$B_{1m} = b_m \bar{U}_\infty, \tag{3.18a}$$

$$A_{jm}\sinh\xi_{jm} + B_{jm}\cosh\xi_{jm} - B_{(j+1)m} = 0, \qquad j = 1, \ldots, n-1, \tag{3.18b}$$

$$A_{jm}\cosh\xi_{jm} + B_{jm}\sinh\xi_{jm} - h_j A_{(j+1)m} = 0, \qquad j = 1, \ldots, n-1, \tag{3.18c}$$

$$A_{nm}\sinh\xi_{nm} + B_{nm}\cosh\xi_{nm} = b_m \bar{U}_\infty. \tag{3.18d}$$

The coefficients A_{jm} and B_{jm} can be solved from equation (3.18) by Gramer's rule as follows:

Let $\Delta_m(s) =$

$$\begin{vmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\sinh\xi_{1m} & \cosh\xi_{1m} & 0 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\cosh\xi_{1m} & \sinh\xi_{1m} & -h_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & \sinh\xi_{2m} & \cosh\xi_{2m} & 0 & -1 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & \cosh\xi_{2m} & \sinh\xi_{2m} & -h_2 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \sinh\xi_{(n-1)m} & \cosh\xi_{(n-1)m} & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \cosh\xi_{(n-1)m} & \sinh\xi_{(n-1)m} & -h_{n-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \sinh\xi_{nm} & \cosh\xi_{nm}
\end{vmatrix} \tag{3.19a}$$

$$A_{jm} = \frac{\begin{array}{c} \text{Column } 2j-1 \\ \begin{vmatrix} \ldots & b_m\bar{U}_\infty & \ldots \\ \ldots & 0 & \ldots \\ \Delta_m(s) & \ldots & \Delta_m(s) \\ \ldots & 0 & \ldots \\ \ldots & b_m\bar{U}_\infty & \ldots \end{vmatrix} \end{array}}{\Delta_m(s)} = \left[\Delta^1_{jm}(s) + \Delta^2_{jm}(s)\right]\bar{U}_\infty, \tag{3.19b}$$

$$B_{jm} = \frac{\begin{array}{c} \text{Column } 2j \\ \downarrow \\ \begin{vmatrix} \dots & b_m \bar{U}_\infty & \dots \\ \dots & 0 & \dots \\ \Delta_m(s) & \dots & \Delta_m(s) \\ \dots & 0 & \dots \\ \dots & b_m \bar{U}_\infty & \dots \end{vmatrix} \end{array}}{\Delta_m(s)} = \left[\Delta^3_{jm}(s) + \Delta^4_{jm}(s)\right]\bar{U}_\infty, \tag{3.19c}$$

where

$$\left.\begin{aligned} \Delta^1_{jm}(s) &= b_m \frac{\begin{vmatrix} \Delta_m(s) & \text{with} \\ \text{row} - 1 & \text{column } -2j-1 \\ \text{deleted} & \end{vmatrix}}{\Delta_m(s)}, \\ \Delta^2_{jm}(s) &= -b_m \frac{\begin{vmatrix} \Delta_m(s) & \text{with} \\ \text{row} - 2n & \text{column } -2j-1 \\ \text{deleted} & \end{vmatrix}}{\Delta_m(s)}, \end{aligned}\right\} \tag{3.19d}$$

$$\Delta^3_{jm}(s) = -b_m \frac{\begin{vmatrix} \Delta_m(s) & \text{with} \\ \text{row} - 1 & \text{column } -2j \\ \text{deleted} & \end{vmatrix}}{\Delta_m(s)}, \qquad \Delta^4_{jm}(s) = b_m \frac{\begin{vmatrix} \Delta_m(s) & \text{with} \\ \text{row} - 2n & \text{column } -2j \\ \text{deleted} & \end{vmatrix}}{\Delta_m(s)}. \tag{3.19e}$$

Thereby, equation (3.17) is obtained as

$$\begin{aligned} \bar{X}_{jm} &= A_{jm} \sinh(q_{jm}(x - L_{j-1})) + B_{jm} \cosh(q_{jm}(x - L_{j-1})) \\ &= \left(\Delta^1_{jm}(s) + \Delta^2_{jm}(s)\right) \sinh(q_{jm}(x - L_{j-1}))\bar{U}_\infty \\ &\quad + \left(\Delta^3_{jm}(s) + \Delta^4_{jm}(s)\right) \cosh(q_{jm}(x - L_{j-1}))\bar{U}_\infty \\ &= F_{jm}(s, x)\bar{U}_\infty, \end{aligned} \tag{3.20a}$$

where

$$\begin{aligned} F_{jm}(s, x) = {} & \left(\Delta^1_{jm}(s) + \Delta^2_{jm}(s)\right) \sinh(q_{jm}(x - L_{j-1})) + \left(\Delta^3_{jm}(s) + \Delta^4_{jm}(s)\right) \\ & \times \cosh(q_{jm}(x - L_{j-1})), x \in [L_{j-1}, L_j], \quad j = 1, \dots, n. \end{aligned} \tag{3.20b}$$

3.3.2. Solutions of the equations. To illustrate the solution method, let $f_{jm}(t, x)$ be the inverse Laplace transform of $F_{jm}(s, x)$. Equation (3.20a) is then expressed as (see (3.16b))

$$X_{jm} = f_{jm}(t, x) * U_\infty(t) = \int_0^t f_{jm}(\tau, x) U_\infty(t - \tau)\, \mathrm{d}\tau = \int_0^\infty - \int_t^\infty f_{jm}(\tau, x) U_\infty(t - \tau)\, \mathrm{d}\tau. \tag{3.21}$$

As $f_{jm}(t, x)$ is a bounded function, the second term of (3.21) will tend to 0 when time is long enough. Therefore,

$$\begin{aligned} X_{jm} &\approx \int_0^\infty f_{jm}(\tau, x) U_\infty(t - \tau)\, \mathrm{d}\tau = \int_0^\infty f_{jm}(\tau, x) \exp(\mathrm{i}\omega(t - \tau) + \mathrm{i}\varphi)\, \mathrm{d}\tau \\ &= \exp(\mathrm{i}\omega t + \mathrm{i}\varphi) \int_0^\infty \underbrace{\exp(-\mathrm{i}\omega\tau) f_{jm}(\tau, x)\, \mathrm{d}\tau}_{\text{Laplace transform of } f_{jm} \text{ at } \mathrm{i}\omega = F_{jm}(\mathrm{i}\omega, x), \text{ see (3.16}a\text{)}} = F_{jm}(\mathrm{i}\omega, x) \exp(\mathrm{i}\omega t + \mathrm{i}\varphi). \end{aligned} \tag{3.22}$$

Note the inverse Laplace transform $f_{jm}(t, x)$ is acting only as a symbolic function. By taking an advantage of the mathematical expression of exponential functions, $f_{jm}(t, x)$ is replaced by its Laplace transform. In this way, complicated residue calculation is avoided.

3.4. Final solution of P1

By equations (3.10) and (3.22), solution to P1 is obtained as

$$U_j^1(t, x, y) = \sum_{m=1}^{\infty} X_{jm}(t, x)\sin(m\pi y) = \sum_{m=1}^{\infty} F_{jm}(\mathrm{i}\omega, x)\sin(m\pi y)\exp(\mathrm{i}\omega t + \mathrm{i}\varphi), \tag{3.23}$$

where $F_{jm}(t, x)$ is given in equation (3.20*b*). Next we need to find the solutions for P2.

4. Analytical solutions for P2

4.1. Simplification of the problem P2

We introduce the new variable

$$u_j = U_j^2 \exp(-\mathrm{i}\omega t - \mathrm{i}\varphi). \tag{4.1}$$

Then P2 becomes

$$P2: \quad \frac{\partial u_j}{\partial t} + \mathrm{i}\omega u_j = k_j \frac{\partial^2 u_j}{\partial x^2} + k_j \frac{\partial^2 u_j}{\partial y^2}, \qquad x \in [L_{j-1}, L_j], \quad y \in [0, 1], \quad j = 1, \ldots, n \tag{4.2a}$$

and boundary conditions are

$$u_1(t, L_0, y) = 0, \qquad y \in [0, 1], \tag{4.2b}$$

$$u_j(t, L_j, y) = u_{j+1}(t, L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{4.2c}$$

$$-\lambda_j \frac{\partial u_j}{\partial x}(t, L_j, y) = -\lambda_{j+1}\frac{\partial u_{j+1}}{\partial x}(t, L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{4.2d}$$

$$u_n(t, L_n, y) = 0, \qquad y \in [0, 1], \tag{4.2e}$$

$$u_j(t, x, 0) = 1, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{4.2f}$$

$$u_j(t, x, 1) = 1, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{4.2g}$$

$$u_j(0, x, y) = 0, \qquad x \in [L_{j-1}, L_j], \quad y \in [0, 1], \quad j = 1, \ldots, n. \tag{4.2h}$$

System (4.2) can be split up into the following two subproblems:

$$u_j(t, x, y) = u_j^1(x, y) + u_j^2(t, x, y), \qquad j = 1, \ldots, n. \tag{4.3}$$

$$P2\text{-}1: \quad k_j \frac{\partial^2 u_j^1}{\partial x^2} + k_j \frac{\partial^2 u_j^1}{\partial y^2} = \mathrm{i}\omega u_j^1, \qquad x \in [L_{j-1}, L_j], \quad y \in [0, 1], \quad j = 1, \ldots, n \tag{4.4a}$$

and boundary conditions are

$$u_1^1(L_0, y) = 0, \qquad y \in [0, 1], \tag{4.4b}$$

$$u_j^1(L_j, y) = u_{j+1}^1(L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{4.4c}$$

$$-\lambda_j \frac{\partial u_j^1}{\partial x}(L_j, y) = -\lambda_{j+1} \frac{\partial u_{j+1}^1}{\partial x}(L_j, y), \qquad y \in [0,1], \quad j = 1, \ldots, n-1, \tag{4.4d}$$

$$u_n^1(L_n, y) = 0, \qquad y \in [0,1], \tag{4.4e}$$

$$u_j^1(x, 0) = 1, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{4.4f}$$

$$u_j^1(x, 1) = 1, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n. \tag{4.4g}$$

$$P2\text{-}2: \quad \frac{\partial u_j^2}{\partial t} + \mathrm{i}\omega u_j^2 = k_j \frac{\partial^2 u_j^2}{\partial x^2} + k_j \frac{\partial^2 u_j^2}{\partial x^2}, \quad x \in [L_{j-1}, L_j], \quad y \in [0,1], \quad j = 1, \ldots, n, \tag{4.5a}$$

and boundary conditions are

$$u_1^2(t, L_0, y) = 0, \qquad y \in [0,1], \tag{4.5b}$$

$$u_j^2(t, L_j, y) = u_{j+1}^2(t, L_j, y), \qquad y \in [0,1], \quad j = 1, \ldots, n-1, \tag{4.5c}$$

$$-\lambda_j \frac{\partial u_j^2}{\partial x}(t, L_j, y) = -\lambda_{j+1} \frac{\partial u_{j+1}^2}{\partial x}(t, L_j, y), \qquad y \in [0,1], \quad j = 1, \ldots, n-1, \tag{4.5d}$$

$$u_n^2(t, L_n, y) = 0, \qquad y \in [0,1], \tag{4.5e}$$

$$u_j^2(t, x, 0) = 0, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{4.5f}$$

$$u_j^2(t, x, 1) = 0, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{4.5g}$$

$$u_j^2(0, x, y) = 0, \qquad x \in [L_{j-1}, L_j], \quad y \in [0,1], \quad j = 1, \ldots, n. \tag{4.5h}$$

It is easily to see that there exists only trivial solution for P2-2, namely:

$$u_j^2 = 0, \qquad j = 1, \ldots, n. \tag{4.6}$$

Therefore, we concentrate on solving problem P2-1 only.

4.2. Analytical solutions to P2-1

For convenience, we put

$$v_j = u_j^1 - 1 \qquad \text{and} \qquad c_{jm} = \frac{k_j m^2 \pi^2}{k_j m^2 \pi^2 + \mathrm{i}\omega}, \qquad j = 1, \ldots, n, \quad m = 1, \ldots, \infty \tag{4.7}$$

and re-write P2-1 as

$$P2\text{-}1: \quad k_j \frac{\partial^2 v_j}{\partial x^2} + k_j \frac{\partial^2 v_j}{\partial y^2} = \mathrm{i}\,\omega(v_j + 1), \qquad x \in [L_{j-1}, L_j], \quad y \in [0,1], \quad j = 1, \ldots, n \tag{4.8a}$$

and boundary conditions are

$$v_1(L_0, y) = -1, \qquad y \in [0,1], \tag{4.8b}$$

$$v_j(L_j, y) = v_{j+1}(L_j, y), \qquad y \in [0,1], \quad j = 1, \ldots, n-1, \tag{4.8c}$$

$$-\lambda_j \frac{\partial v_j}{\partial x}(L_j, y) = -\lambda_{j+1} \frac{\partial v_{j+1}}{\partial x}(L_j, y), \qquad y \in [0, 1], \quad j = 1, \ldots, n-1, \tag{4.8d}$$

$$v_n(L_n, y) = -1, \qquad y \in [0, 1], \tag{4.8e}$$

$$v_j(x, 0) = 0, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n, \tag{4.8f}$$

$$v_j(x, 1) = 0, \qquad x \in [L_{j-1}, L_j], \quad j = 1, \ldots, n. \tag{4.8g}$$

The solutions are separated by assuming a product of the form

$$v_j(x, y) = X_j(x) Y_j(y). \tag{4.9}$$

Let Y_j be such equation which satisfies the homogeneous equation of (4.8a), then

$$\frac{Y_j''}{Y_j} = -\mu_j^2, \tag{4.10}$$

Boundary equations (4.8f), (4.8g) give

$$\begin{aligned} &\mu_j = m\pi \quad \text{or} \quad \mu_{jm} = \mu_m = m\pi \qquad \text{and} \\ &Y_{jm}(y) = Y_m(y) = \sin(m\pi y), \quad m = 1, \ldots, \infty. \end{aligned} \tag{4.11}$$

Hence

$$v_j(x, y) = \sum_{m=1}^{\infty} X_{jm}(x) Y_m(y). \tag{4.12}$$

Inserting equation (4.12) into (4.8a) and noting that $1 = \sum b_m Y_m$ we get

$$\begin{aligned} &k_j \sum_{m=1}^{\infty} X_{jm}''(x) Y_m(y) - k_j \sum_{m=1}^{\infty} m^2\pi^2 X_{jm}(x) Y_m(y) \\ &\qquad = \mathrm{i}\omega \sum_{m=1}^{\infty} X_{jm}(x) Y_m(y) + \mathrm{i}\omega \sum_{m=1}^{\infty} b_m Y_m(y). \end{aligned} \tag{4.13}$$

The orthogonal properties of $Y_m(y) = \sin(m\pi y)$ ensure that

$$k_j X_{jm}'' - k_j m^2\pi^2 X_{jm} = \mathrm{i}\omega X_{jm} + \mathrm{i}\omega b_m. \tag{4.14}$$

By setting $q_{jm} = \sqrt{\frac{\mathrm{i}\omega}{k_j} + m^2\pi^2}$, solution of $X_{jm}(x)$ in equation (4.14) can be expressed as

$$X_{jm} = A_{jm} \sinh(q_{jm}(x - L_{j-1})) + B_{jm} \cosh(q_{jm}(x - L_{j-1})) - (1 - c_{jm}) b_m. \tag{4.15}$$

So the general solution of P2-1 can be expressed as a linear combination of these solutions, namely

$$v_j(x, y) = \sum_{m=1}^{\infty} X_{jm}(x) \sin(m\pi y), \qquad j = 1, \ldots, n. \tag{4.16}$$

Boundary conditions (4.8b) and (4.8e) provide that

$$-1 = v_1(L_0, y) = \sum_{m=1}^{\infty} \big(A_{1m} \sinh(q_{1m} L_0) + B_{1m} \cosh(q_{1m} L_0) - (1 - c_{1m}) b_m\big) \sin(m\pi y). \tag{4.17}$$

Using the orthogonal property of the function $\sin(m\pi y)$, we get

$$B_{1m} = -c_{1m} b_m. \tag{4.18a}$$

Similarly,

$$A_{nm}\sinh(q_{nm}L_n) + B_{nm}\cosh(q_{nm}L_n) = -c_{nm}b_m. \tag{4.18b}$$

Together with the boundary conditions (4.8*c*) and (4.8*d*) we get the following linear system equations for the coefficients A_{jm} and B_{jm}, $j = 1, \ldots, n$,

$$B_{1m} = -c_{1m}b_m, \tag{4.19a}$$

$$A_{jm}\sinh\xi_{jm} + B_{jm}\cosh\xi_{jm} - B_{(j+1)m} = 0, \qquad j = 1, \ldots, n-1, \tag{4.19b}$$

$$A_{jm}\cosh\xi_{jm} + B_{jm}\sinh\xi_{jm} - h_j A_{(j+1)m} = 0, \qquad j = 1, \ldots, n-1, \tag{4.19c}$$

$$A_{nm}\sinh\xi_{nm} + B_{nm}\cosh\xi_{nm} = -c_{nm}b_m, \tag{4.19d}$$

where

$$q_{jm} = \sqrt{\frac{\mathrm{i}\omega}{k_j} + m^2\pi^2}, \qquad \xi_{jm} = q_{jm}l_j, \quad j = 1, \ldots, n, \quad h_j = \frac{\lambda_{j+1}}{\lambda_j}\sqrt{\frac{k_j}{k_{j+1}}},$$
$$j = 1, \ldots, n-1, \quad m = 1, \ldots, \infty. \tag{4.19e}$$

Symbolically, the calculations follow exactly the same steps as those in section 3.3.1 by setting $s = \mathrm{i}\omega$ for the *j*th layer. Therefore, equations (3.19) give

$$A_{jm} = \frac{\begin{array}{c} \downarrow \ \text{Column } 2j-1 \\ \begin{vmatrix} \cdots & -c_{1m}b_m & \cdots \\ \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots \\ \cdots & -c_{nm}b_m & \cdots \end{vmatrix} \end{array}}{\Delta_m(\mathrm{i}\omega)} = -c_{1m}\Delta_{jm}^1(\mathrm{i}\omega) - c_{nm}\Delta_{jm}^2(\mathrm{i}\omega), \tag{4.20a}$$

$$B_{jm} = \frac{\begin{array}{c} \downarrow \ \text{Column } 2j \\ \begin{vmatrix} \cdots & -c_{1m}b_m & \cdots \\ \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots \\ \cdots & -c_{nm}b_m & \cdots \end{vmatrix} \end{array}}{\Delta_m(\mathrm{i}\omega)} = -c_{1m}\Delta_{jm}^3(\mathrm{i}\omega) - c_{nm}\Delta_{jm}^4(\mathrm{i}\omega). \tag{4.20b}$$

Hence, from equations (4.15), (4.16)

$$v_j(x, y) = \sum_{m=1}^{\infty} X_{jm}(x)\sin(m\pi y) = \sum_{m=1}^{\infty} G_{jm}(\mathrm{i}\omega, x)\sin(m\pi y), \qquad j = 1, \ldots, n, \tag{4.21a}$$

where

$$G_{jm}(\mathrm{i}\omega, x) = -\left[c_{1m}\Delta_{jm}^1(\mathrm{i}\omega) + c_{nm}\Delta_{jm}^2(\mathrm{i}\omega)\right]\sinh(q_{jm}(x - L_{j-1}))$$
$$- \left[c_{1m}\Delta_{jm}^3(\mathrm{i}\omega) + c_{nm}\Delta_{jm}^4(\mathrm{i}\omega)\right]\cosh(q_{jm}(x - L_{j-1})) - (1 - c_{jm})b_m. \tag{4.21b}$$

Finally, from equations (4.1), (4.3) and (4.7), the solution of P2 can be obtained as

$$U_j^2 = (v_j + 1)\exp(\mathrm{i}\omega t + \mathrm{i}\varphi), \qquad j = 1, \ldots, n. \tag{4.22}$$

Table 1. Material properties of the composite slab.

Material	Thermal conductivity ($W\ m^{-1}\ K^{-1}$)	Thermal diffusivity ($m^2\ s^{-1}$)	Thickness (mm)
1. Wall paper	0.12	1.5×10^{-7}	25
2. Mineral wool	0.0337	1.47×10^{-6}	200
3. Concrete	0.9	3.75×10^{-7}	100
4. Plywood	0.147	1.61×10^{-7}	100
5. Gypsum board	0.23	4.11×10^{-7}	13

4.3. Final solutions

From equation (2.3) the solutions to heat conduction for a two-dimensional composite slab are obtained explicitly as (see (2.3), (3.23) and (4.22)).

$$T_j(t,x,y) = \mathrm{real}\left\{\left[\sum_{m=1}^{\infty}(F_{jm}(\mathrm{i}\omega,x)+G_{jm}(\mathrm{i}\omega,x))\sin(m\pi y)+1\right]\exp(\mathrm{i}\omega t+\mathrm{i}\varphi)\right\},\quad j=1,\ldots,n, \tag{4.23}$$

where F_{jm} and G_{jm} are defined in equations (3.20*b*) and (4.21*b*).

5. Solutions for more general boundary conditions

For completeness, we demonstrate the solution in the case of more general boundary conditions without showing all the details. The Fourier series of time-dependent boundaries take the form

$$T_\infty(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\omega_k t + \varphi_k). \tag{5.1}$$

The problem can be split up into two subproblems with boundary conditions which are constant a_0 and the periodic change $\sum_{k=1}^{\infty} a_k \cos(\omega_k t + \varphi_k)$. The solution in the second subproblem can be obtained in equation (4.23) with boundary $\sum_{k=1}^{\infty} a_k \exp(\mathrm{i}\omega_k t + \mathrm{i}\varphi_k)$.

For the first subproblem with constant boundary condition, if we ignore the transient term which will eventually die away, the temperature distribution is the solution of the steady-state situation a_0.

6. Calculation example

We focus on the demonstration of the analytical solutions. A selected five-layer slab is presented here to assess the accuracy of the method by comparisons with numerical solutions. The slab is an extension of a three-layer wall structure used in our test building whose material and physical properties are listed in table 1. The foregoing conditions are shown schematically in figure 2.

The first boundary conditions were taken from measured monthly weather statistics from 1971 to 2000 in Helsinki area [14]. The yearly statistics data were fitted with cosine function with period 365 days: $T_\infty(t) = 5.6 - 10.7\cos\left(\frac{2\pi}{365}(t-20.0)\right)$.

Calculation of transient temperature change was made over the central region of the slab in figure 2 for materials 2 and 3. Figures 3 and 4 show the comparisons of the transient temperature variations by using the analytical and numerical methods. The temperatures were

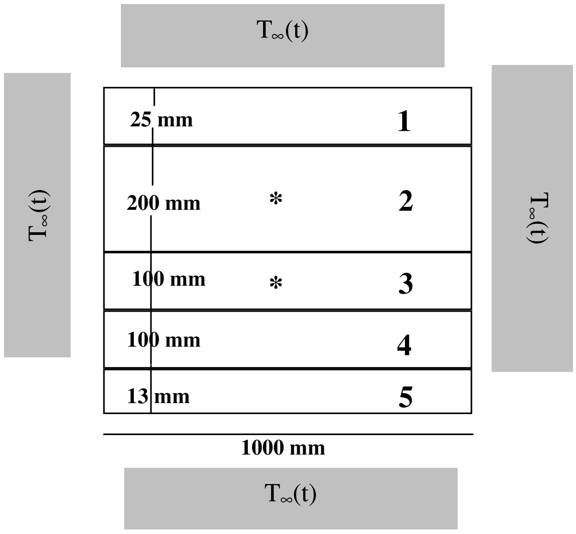

Figure 2. Schematic diagram of the five-layer composite slab.

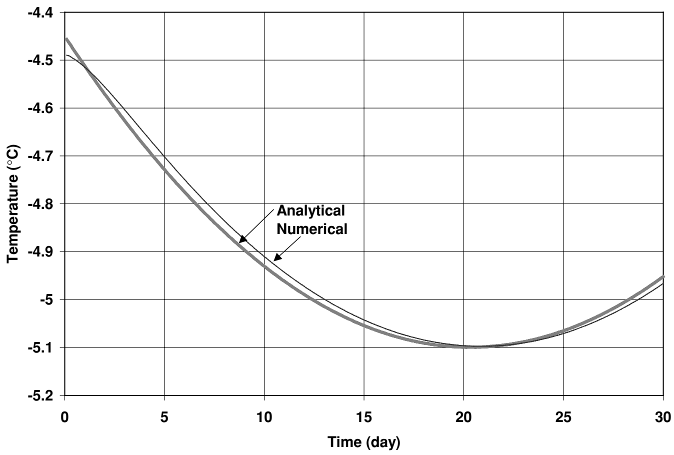

Figure 3. Comparison of analytical and numerical results at the point marked with star in material 2.

stored in files as hourly values and shown in figures as hourly and daily values. In both demonstrated points, the maximal discrepancy is about 0.12 °C with relative error of 3%. The initial value was roughly estimated with numerical programme.

The second boundary was taken from measurements and then fitted with cosine functions with periods 120, 30, 10, 5 and 1 days. The cosine functions were of the following type:

$$T_\infty(t) = a_0 + \sum_{1}^{5} a_i \cos\left(\frac{2\pi t}{\omega_i} - \varphi_i\right), \tag{6.1}$$

where the fitting parameters are listed in table 2.

The calculated point was marked with a star in material 3. The transient temperature change is displayed in figure 5. The biggest discrepancy is about 0.28 °C with relative error less than 2% for the analytical and numerical results. As in the previous case, the initial value

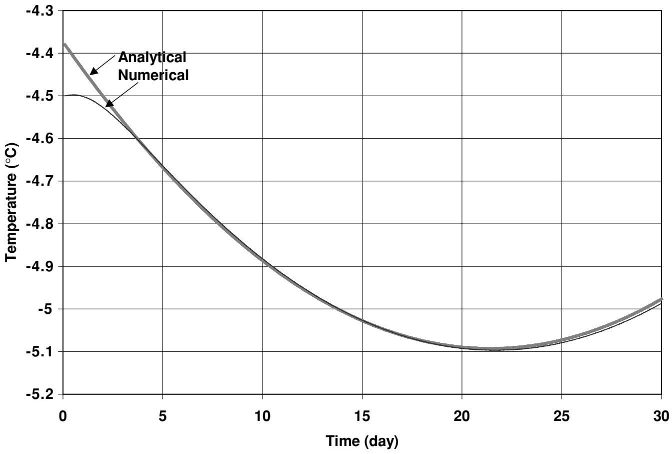

Figure 4. Comparison of analytical and numerical results at the point marked with a star in material 3.

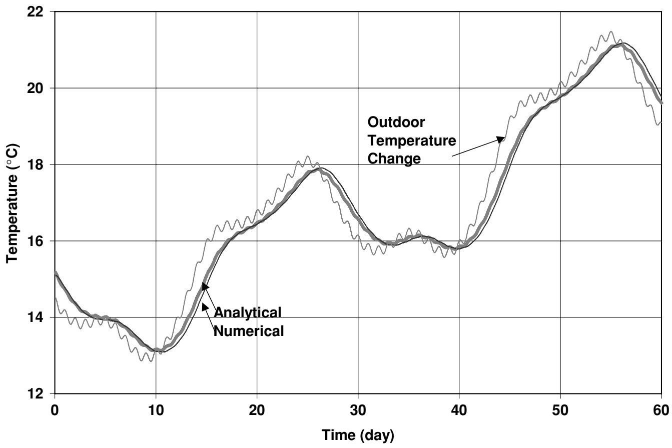

Figure 5. Comparison of analytical and numerical results at the point marked with a star in material 3.

Table 2. Parameters of equation (6.1).

	ω_1	ω_2	ω_3	ω_4	ω_5
	120.0	30.0	10.0	5.0	1.0
	φ_1	φ_2	φ_3	φ_4	φ_5
	28.566 93	7.790 886	6.483 743	3.647 276	3.295 951
a_0	a_1	a_2	a_3	a_4	a_5
17.298 92	2.3712	−1.774 64	−0.533 07	−0.053 64	−0.121 07

was roughly estimated with the numerical programme. The calculation shows a good accuracy of the method. The validation of the numerical programme can be found in [15–17]. More calculations for temperature distributions also demonstrated a good accuracy of the method without showing any substantial difference.

7. Discussions

7.1. Calculation steps

Based on the above statements, we summarize the calculation process as follows:

Step 1. Give λ_j, k_j and l_j, $j = 1, \ldots, n$ for an n-layer composite slab. Outdoor temperature is approximated as $T_\infty(t) = a_0 + \sum_{i=1}^{N1} a_i \cos(\omega_i t + \varphi_i)$. For any point (x, y) at the jth layer, go to step 2.

Step 2. For $i = 1$ to N1, calculate C_i: from $m = 1$ to N2 (N2 = 100 for example),

$$b_m = \frac{2 - 2\cos(m\pi)}{m\pi}, \qquad q_m = \sqrt{\frac{\mathrm{i}\omega_i}{k_j} + m^2\pi^2}, \qquad c_{1m} = \frac{k_1 m^2\pi^2}{k_1 m^2\pi^2 + \mathrm{i}\omega_i},$$

$$c_{jm} = \frac{k_j m^2\pi^2}{k_j m^2\pi^2 + \mathrm{i}\omega_i}, \qquad c_{nm} = \frac{k_n m^2\pi^2}{k_n m^2\pi^2 + \mathrm{i}\omega_i},$$

$\Delta_m(\mathrm{i}\omega_i)$, $\Delta_m^1(\mathrm{i}\omega_i)$, $\Delta_m^2(\mathrm{i}\omega_i)$, $\Delta_m^3(\mathrm{i}\omega_i)$, $\Delta_m^4(\mathrm{i}\omega_i)$, F_m (equation (3.20b)), G_m (equation (4.21b)), then $C_i = \sum_{m=1}^{N2} (F_m + G_m) \sin(m\pi y)$. Go to step 3.

Step 3. Compute $\sum_{i=1}^{N1} a_i C_i$, C_i is obtained from step 2. The final solution is

$$a_0 + \mathrm{real}\left\{\left(\sum_{i=1}^{N1} a_i C_i + 1\right) \exp(\omega_i t + \varphi_i)\right\}.$$

7.2. Some observations

From the above calculation steps, we make some observations.

- The calculation includes only simple computation of matrix determinant which can be easily accomplished by commercial mathematical packages such as Maple, Matlab and Mathematica for example. No numerical work is necessary. For any jth layer, only five sparse matrices are involved. The calculation load is small and the computing time is short.
- Compared with numerical methods, the developed method is easier to implement and a possible instability in numerical methods is avoided. The accuracy is good. Note that there exists a restriction on time step as a function of mesh size in numerical methods.
- With a periodic excitation of boundary conditions, the temperature variation of any jth layer slab is expressed as periodic excitation with attenuated amplitudes and shifted phases which are given in equation (4.23) for example:

$$\begin{aligned} T_j(t, x, y) &= \mathrm{real}\left\{\left[\sum_{m=1}^{\infty} (F_{jm}(\mathrm{i}\omega, x) + G_{jm}(\mathrm{i}\omega, x)) \sin(m\pi y) + 1\right] \exp(\mathrm{i}\omega t + \mathrm{i}\varphi)\right\} \\ &= \left\|\left[\sum_{m=1}^{\infty} (F_{jm}(\mathrm{i}\omega, x) + G_{jm}(\mathrm{i}\omega, x)) \sin(m\pi y) + 1\right]\right\| \cos(\omega t + \mathrm{i}\varphi + \mathrm{d}\varphi_j). \end{aligned}$$

  The time lag is $\mathrm{d}\varphi_j$ which can be estimated from F and G.
- Likewise, F and G can be expressed as algebraic functions of any jth layer's physical properties k_j for instance. Hence the effect on the solution of different physical properties is clearly shown. Due to the space limit, we are not going to proceed with this analysis.

8. Conclusions

In this paper, an analytical approach has been presented for multi-dimensional heat conduction in a composite slab subject to time-dependent temperature changes. The applied technique of '*separation of variable*s' is new. The benefit of the result is its simple and concise mathematical forms of the solutions which can be used to analyse attenuated amplitudes and shifted phases in combination with material properties in the heat transfer process. The physical parameters are clearly shown in the solution formula. Agreement with numerical solutions is good. In a general conduction or diffusion application context, however, numerical schemes have usually been necessary. The proposed approach is free of these restrictions.

One conclusion to be drawn is the accuracy of the analytical solutions, as an approximation formula was used in deriving solutions. Comparing analytical and numerical results shows that analytical solutions are accurate.

It is known that any periodic and piecewise continuous function can be approximated as its Fourier expansion. Therefore, the solution obtained in this paper has a very broad application range.

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